

Potts model on complex networks

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Abstract. We consider the general p -state Potts model on random networks with a given degree distribution (random Bethe lattices). We find the effect of the suppression of a first order phase transition in this model when the degree distribution of the network is fat-tailed, that is, in more precise terms, when the second moment of the distribution diverges. In this situation the transition is continuous and of infinite order, and size effect is anomalously strong. In particular, in the case of $p = 1$, we arrive at the exact solution, which coincides with the known solution of the percolation problem on these networks.

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1 Introduction

Complex networks display a spectrum of unique effects [1–9]. Cooperative phenomena in complex networks are attracting much attention these days. The theoretical study of various cooperative models on random networks [10–23] has demonstrated that their critical behaviour is extremely far from that on regular lattices and on ‘planar graphs’ [24].

The percolation problem on uncorrelated networks with a given degree distribution has been studied in references [18–20]. Here, *degree* is the number of connections of a vertex. The Ising model on these networks was studied by simulations in reference [10] and solved in references [11,12] (see also Ref. [17]). It was shown that the presence of fat tails in the degree distributions of networks dramatically changes the critical behaviour of these basic cooperative models. But the site percolation problem and the Ising model are only particular cases of the general p -state Potts model [25,26]. The site percolation is equivalent to the one-state Potts model [27], and the Ising model is exactly the two-state Potts model. At $p \geq 3$, the Potts model shows features, very different from those at $p = 1$ and $p = 2$. In standard mean-field theory, the $p \geq 3$ -state Potts model has a first-order phase transition in contrast to percolation and the Ising model, where the phase transitions are of second order.

Thus, the Potts model provides an essentially more wide range of behaviours than the percolation problem and the Ising model. The Potts model is related to a

number of outstanding problems in statistical and mathematical physics, and in graph theory, e.g., the colouring graph problem, etc. (for numerous applications of the Potts model see references [25,26]). In short, this is a basic model of statistical mechanics, the direct generalization of the percolation problem and the Ising model. Consequently, before more complicated cooperative models on networks, one has to solve the Potts model at least on the simplest complex networks.

In the present paper we report our exact and asymptotically exact results for the thermodynamic properties of the p -state Potts model on uncorrelated random networks with a given degree distribution. These networks are the undirected graphs, maximally random (i.e., with maximum entropy) under the constraint that their degree distribution is a given one, $P(k)$. Correlations between degrees of vertices in such graphs are absent, as well as clustering. In graph theory, these networks are called ‘labeled random graphs with a given degree sequence’ or ‘the configuration model’ [28]. One should stress that this is the minimal model of complex networks. Most of results on cooperative models on networks were obtained just for this basic construction [18–21,23,29–32]. See, however, references [33–36], where the Berezinski-Kosterlitz-Thouless percolation phase transition was studied in growing networks. See also reference [37] and references therein for cooperative models on networks with correlations between degrees of the nearest-neighbour vertices.

In this paper we demonstrate a strong effect of a fat tail in the degree distribution of a network on a first-order phase transition which occurs in the Potts model with

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$p \geq 3$. We observe the suppression of the first-order phase transition in networks with a fat-tailed degree distribution. These results were announced in reference [22].

It is convenient to use the power-law degree distribution $P(k) \propto k^{-\gamma}$ for parametrization. Then, $\langle k^4 \rangle$ diverges for $\gamma \leq 5$, $\langle k^3 \rangle$ diverges for $\gamma \leq 4$, and $\langle k^2 \rangle$ diverges for $\gamma \leq 3$.

We find that if $p \geq 3$, the first order phase transition occurs only for $\gamma > 3$, while for $\gamma \leq 3$ the phase transition is continuous. At $\gamma = 3$, the phase transition is of infinite order and similar to the transition in the Ising model on these networks [11]. We also obtain the exact solution of the one-state Potts model on networks and show that it agrees with the solution of the percolation problem on networks [20].

2 Characteristic temperature

We consider large networks with a tree-like ‘local structure’. In other words, any finite environment of a vertex in the infinite network looks like a tree. This structure is a property of large uncorrelated networks. Interactions are transmitted through edges, from vertex to vertex. Thus, it is the distribution of the number of connections of a nearest neighbour of a vertex that is crucial. In the uncorrelated random networks, this distribution is $kP(k)/\langle k \rangle$. Then the nearest neighbours of a vertex have the average number of connections (the average degree) $\langle k^2 \rangle / \langle k \rangle$, its second nearest neighbours have the same average degree, and so on. This value is greater than the average number of connections for the entire network $\langle k \rangle$, and it is much greater than $\langle k \rangle$ if $\langle k^2 \rangle$ is large. It is not the mean degree of a network that determines its cooperative behaviour but, rather, the average degree of the nearest neighbour of a vertex [11]. This enables us to estimate a certain characteristic temperature of the p -state Potts model on the network using the formula $T_c/J = 1/\ln[(q+p-2)/(q-2)]$ for the p -state Potts model on a regular Cayley tree with the coordination number $q = \langle k^2 \rangle / \langle k \rangle$ [26]. The result is

$$\frac{J}{T_c} = \ln \frac{\langle k^2 \rangle + (p-2)\langle k \rangle}{\langle k^2 \rangle - 2\langle k \rangle}. \quad (1)$$

This naive estimate is exact for the Ising model [11]. It also gives the exact percolation threshold as we will show below. Note, however, that the meaning of the critical temperature T_c is different for $p = 1, 2$ and $p \geq 3$. At $p = 1$ the parameter $q_c = \exp(J/T_c) - 1$ determines the percolation threshold on networks (q is the probability that a vertex is retained). At $p = 2$, T_c is the critical temperature of the continuous phase transitions in the Ising model on networks (with J replaced by $2J$). We shall show that for $p \geq 3$ and $\gamma > 3$, the p -state Potts model undergoes a first order phase transition, and T_c given by equation (1) is the low-temperature boundary of the region of hysteresis phenomena. We shall show, however, that if $\gamma \leq 3$ and $p \geq 1$, the transition in the Potts model becomes continuous, and T_c is again the critical temperature of the transition (see discussion below).

3 General approach

Consider the p -state Potts model with the Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\alpha_i, \alpha_j} - H \sum_i \delta_{\alpha_i, 1}, \quad (2)$$

where the first sum is over all edges of the graph, the second one is over all vertices. $\delta_{\alpha, \beta} = 0, 1$ if $\alpha \neq \beta$ and $\alpha = \beta$, respectively. Each vertex i can be in any of p states, i.e., $\alpha_i = 1, 2, \dots, p$. We assume a ‘ferromagnetic’ interaction between the nearest-neighbouring vertices, i.e., $J > 0$. The ‘magnetic field’ $H > 0$ distinguishes the state $\alpha = 1$. Hereafter, we set $J = 1$. The p -state Potts model on the regular Cayley tree is solved exactly by using recurrence relations [26]. As networks under discussion have a local tree-like structure, we apply the method of recurrence relations to our problem. Actually, we use the same approach as in our solution of the Ising model on networks [11].

Consider a vertex 0 with k_0 adjacent vertices with ‘spins’ in states $\alpha_{1,i}$, $i = 1, 2, \dots, k_0$. Due to the local tree-like structure, this vertex may be treated as a root of a tree. We introduce

$$g_{1,i}(\alpha_0) = \sum_{\{\alpha_l\}} \exp \left[\left(\sum_{\langle lm \rangle} \delta_{\alpha_l, \alpha_m} + \delta_{\alpha_0, \alpha_{1,i}} + H \sum_l \delta_{\alpha_l, 1} \right) / T \right]. \quad (3)$$

The labels l and m run only over vertices that belong to sub-trees with the root vertex $\alpha_{1,i}$, including this vertex. Then the partition function is

$$Z = \sum_{\alpha_0} e^{H\delta_{\alpha_0, 1}/T} \prod_{i=1}^{k_0} g_{1,i}(\alpha_0). \quad (4)$$

Let

$$x_{1,i}(\alpha) \equiv g_{1,i}(\alpha) / g_{1,i}(1), \quad (5)$$

then the ‘magnetic moment’ M of the vertex 0 is

$$\begin{aligned} M &= \frac{p}{p-1} \left\langle \delta_{\alpha_0, 1} - \frac{1}{p} \right\rangle_T \\ &= \frac{1}{p-1} \frac{(p-1)e^{H/T} - \sum_{\alpha \neq 1} \prod_{i=1}^{k_0} x_{1,i}(\alpha)}{e^{H/T} + \sum_{\alpha \neq 1} \prod_{i=1}^{k_0} x_{1,i}(\alpha)}, \end{aligned} \quad (6)$$

where $\langle \dots \rangle_T$ is the thermodynamic average. The parameters $x_{1,i}$ describe the effects of the nearest neighbours on the vertex 0. In turn, $x_{1,i}$ are expressed in terms of parameters $x_{2,l}(\alpha) = g_{2,l}(\alpha) / g_{2,l}(1)$, $l = 1, 2, \dots, k_{1,i}$, which describe effects of vertices in the second shell on vertices in the first shell, and so on. The following recurrence relation

between $x_{n,j}(\alpha)$ and $x_{n+1,l}(\alpha)$ holds at $n \geq 1$ and $\alpha \geq 2$:

$$x_{n,j}(\alpha) = \frac{\left[e^{H/T} + e^{1/T} \prod_{l=1}^{k_{n,j}-1} x_{n+1,l}(\alpha) + \sum_{\beta \neq 1, \alpha} \prod_{l=1}^{k_{n,j}-1} x_{n+1,l}(\beta) \right]}{\left[e^{(1+H)/T} + \sum_{\beta \neq 1} \prod_{l=1}^{k_{n,j}-1} x_{n+1,l}(\beta) \right]}. \quad (7)$$

If a vertex $n+1, l$ is a dead end, then $x_{n+1,l}(\alpha) = 1$ at all α . Deriving the recurrence relations, we started from some vertex 0 and then made the recurrence steps along sub-trees. While solving the recurrence relations (7), we start from distant vertices, i.e., from large n , and descend along sub-trees to the vertex. Note that all states with the index $\alpha \geq 2$ are equivalent in respect of statistics. Only the state $\alpha = 1$ is distinguished by the applied field H .

The recurrence steps equation (7) converge exponentially quickly to the fixed point which does not depend on α . This enables us to put $x_{n,j}(\alpha) = x_{n,j}$ in equation (7) from the very beginning for the sake of simplicity. Then equations (6) and (7) take a form:

$$M = \frac{e^{H/T} - \prod_{i=1}^{k_0} x_{1,i}}{e^{H/T} + (p-1) \prod_{i=1}^{k_0} x_{1,i}}, \quad (8)$$

$$x_{n,j} = y \left(\prod_{l=1}^{k_{n,j}-1} x_{n+1,l} \right), \quad (9)$$

where we introduce

$$y(x) = \frac{e^{H/T} + (e^{1/T} + p - 2)x}{e^{(1+H)/T} + (p-1)x}. \quad (10)$$

Note that at $H = 0$, in the paramagnetic phase $x_{n,l} = 1$ while at $H > 0$, $x_{n,l} \leq 1$. We stress that equations (8) and (9) are exact for any tree-like graph.

4 Exact solution at $p = 1$: percolation on networks

The one-state Potts model on networks at $H = 0$ is of a special interest, as it relates to percolation on networks. This limiting case can be solved exactly for an uncorrelated random graph with an arbitrary degree distribution. For this let us consider the recurrence relation (9) at $p = 1$ and $H = 0$:

$$x_{n,j} = e^{-1/T} + \left(1 - e^{-1/T}\right) \prod_{l=1}^{k_{n,j}-1} x_{n+1,l}. \quad (11)$$

Note that in an uncorrelated random graph, the degrees $k_{n,j}$ are independent random variables. Since $x_{n+1,l}$ in this equation depends only on $k_{m,j}$ with $m \geq n+1$,

one can average the left and right sides of the equation over the ensemble of random graphs with a degree distribution $P(k)$ and find the self-consistent equation for the average $\langle x_{n,j} \rangle$. In the limit $n \rightarrow \infty$, the fixed point $\langle x_{n,j} \rangle \rightarrow x$ of the exact recurrence relation is given by

$$x = 1 - q + \frac{q}{\langle k \rangle} \sum_k P(k) k x^{k-1}, \quad (12)$$

where we introduce the parameter

$$q = 1 - e^{-1/T}. \quad (13)$$

It is important to note that equation (13) establishes the relation between the one-state Potts model and site percolation on uncorrelated networks. The latter is described by equation (12) [20]. The parameter q has the meaning of the retained fraction of vertices. The critical temperature T_c , equation (1), at $p = 1$, determines the percolation threshold $q_c = \langle k \rangle / (\langle k^2 \rangle - \langle k \rangle)$. The strong influence of the fat tail of the degree distribution on percolation critical exponents has been revealed and studied in detail in reference [20].

5 First order phase transition

It is well known that the p -state Potts model, in the framework of the standard mean-field theory, undergoes a first order phase transition for all $p \geq 3$ [25,38]. The general approach derived above enables us to consider, at $H = 0$, the influence of the fat tail of the degree distribution on the transition.

At $p \geq 3$, in order to average over the ensemble of random graphs with a given degree distribution we use the effective medium approach developed in reference [11] for the Ising model on networks. First at all, as at $H \geq 0$ we have $x_{n,l} \leq 1$, it is convenient to introduce positive parameters $h_{n,l}$: $x_{n,l} = \exp(-h_{n,l})$. They are independent random parameters and may be considered as random effective fields acting on a vertex in the n -th shell from neighbouring ‘spins’ in the $n+1$ -th shell. Then, equations (8) and (9) take the form:

$$M = \frac{e^{H/T} - \exp\left(-\sum_{l=1}^{k_0} h_{1,l}\right)}{e^{H/T} + (p-1) \exp\left(-\sum_{l=1}^{k_0} h_{1,l}\right)}, \quad (14)$$

$$h_{n,j} = -\ln \left\{ y \left[\exp\left(-\sum_{l=1}^{k_{n,j}-1} h_{n+1,l}\right) \right] \right\}. \quad (15)$$

At dead ends we have $h_{n+1,l} = 0$. At $H = 0$ in the paramagnetic phase $h_{n,l} = 0$, while in the ordered phase $h_{n,l} \neq 0$. Equations (14) and (15) determine the magnetization M of a graph as a function of T and H .

While solving the recurrence relations (15), we start from distant spins with $h \approx 0$ and descend along sub-trees to α . In the limit $n \rightarrow \infty$, the parameter $h_{1,i}$ is the

fixed point of the recurrence steps. The thermodynamic behaviour is determined by this fixed point.

The right-hand sides of equations (14) and (15) depend only on the sum of the independent and equivalent random variables $h_{n,j}$. So, let us use the following ansatz [11]:

$$\sum_{l=1}^k h_{n,l} \approx kh + \mathcal{O}(k^{1/2}), \quad (16)$$

where $h \equiv \langle h_{n,l} \rangle$ is the average value of the ‘effective field’ acting on a vertex. This approximation takes into account the most ‘dangerous’ highly connected vertices in the best way. With this ansatz, averaging over the ensemble of random graphs and applying the ansatz (16) to equations (14) and (15), we obtain

$$\langle M \rangle = \sum_k P(k) \frac{e^{H/T} - e^{-kh}}{e^{H/T} + (p-1)e^{-kh}}, \quad (17)$$

$$h = -\langle k \rangle^{-1} \sum_k P(k) k \ln y \left[e^{-(k-1)h} \right] \equiv G(h). \quad (18)$$

The parameter h plays the role of the order parameter. At $H = 0$, $h = 0$ above T_c and $h > 0$ below T_c .

Let us describe the thermodynamic properties of the Potts model with $p \geq 3$ on the infinite networks at $H = 0$. For this, one must solve the equation of state (18).

At first, we consider the case $\langle k^4 \rangle < \infty$. The character of the transition and a characteristic temperature may be found from the analysis of the expansion of $G(h)$ over small h . If $p \geq 3$ it is enough to take into account only first two terms: $G(h) = g_1 h + g_2 h^2 + \dots$ where

$$g_1 = \frac{\langle k(k-1) \rangle (e^{1/T} - 1)}{\langle k \rangle (e^{1/T} + p - 1)}, \quad (19)$$

$$g_2 = \frac{\langle k(k-1)^2 \rangle (e^{1/T} - 1) (p-2)}{2 \langle k \rangle (e^{1/T} + p - 1)^2}. \quad (20)$$

One sees that at high temperatures, the coefficient $g_1 \approx 0$. When temperature decreases, g_1 increases and becomes larger than 1. The point where $g_1 = 1$ is the specific point. The corresponding temperature T_c is given by equation (1). As $g_2 > 0$ at all temperatures, the Potts model with $p \geq 3$ undergoes a first order phase transition from the paramagnetic phase to the ordered one. Note that for the Ising model ($p = 2$) the coefficients $g_2 = 0$ and $g_3 < 0$, and the phase transition is continuous [11]. In the paramagnetic phase the probabilities to find any vertex in the states $\alpha = 1, 2, \dots, p$ are equal, which corresponds to the solution $h = 0$. In the ordered phase, one of the states, $\alpha = 1$ in our consideration, has a larger probability in comparison to other states with $\alpha \geq 2$. This corresponds to the solution with $h > 0$.

At large γ , the system undergoes a standard first order phase transition (see the temperature behaviour of M in Figure 1 for $\gamma = 6$). The decrease of γ modifies the second term in the expansion of $G(h)$: at $\gamma = 4$ we have $\langle k^4 \rangle \rightarrow \infty$ and $G(h) = g_1 h + g_2' h^2 \ln(1/h) + \dots$, at $3 < \gamma < 4$

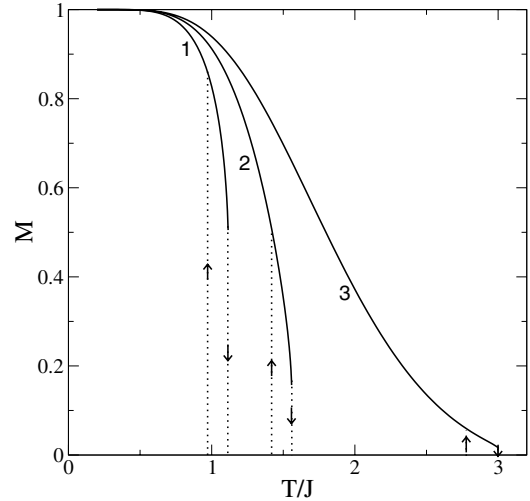


Fig. 1. The temperature dependence of the magnetization of the p -state Potts model with $p = 5$ on uncorrelated random networks with the degree distribution $P(k) = Ak^{-\gamma}$ for different exponents γ : 1) $\gamma = 6$; 2) $\gamma = 4$; 3) $\gamma = 3.2$. The temperature region between dotted lines for each curve is the hysteresis region. The up and down arrows correspond cooling and heating, respectively.

we find $G(h) = g_1 h + g_2' h^{\gamma-2} + \dots$. It is important to note that the coefficient $g_2' > 0$. In a general form, the condition of a first-order phase transition may be written as $d(h^{-1}G)/dh > 0$. We have found that in the range $3 < \gamma \leq 4$ the derivative $d(h^{-1}G)/dh$ at small h is positive and the transition is still of the first order (see in Fig. 1 our results for $\gamma = 4$ and 3.2).

In order to find the temperature behaviour of the magnetization in the first order phase transition, we have solved numerically the equation (18). Results of the numerical solution of equations (18) and (17) reveals that when γ approaches 3 from above, the jump of the magnetic moment in the first order phase transition tends to zero. Moreover, it is interesting to note that the temperature behaviour of M in a wide temperature range just after the jump follows the exponential law $M \sim \exp(-cT)$ with the constant c which depends on the complete degree distribution $P(k)$ (see the results in Fig. 1 for $\gamma = 3.2$). It is the behaviour that is expected for the infinite order phase transition at $\gamma = 3$ [11].

In the case of the first order phase transition, there is a temperature region, $T_2 < T < T_1$, of hysteresis phenomena owing to the existence of metastable states. For the Potts model, the low temperature boundary T_2 is the temperature below which the ordered state with $h \neq 0$ is the only stable state. In other words, below T_2 the free energy as a function of the order parameter has only one minimum corresponding to $h \neq 0$, while the solution $h = 0$ corresponds to the maximum. One can show that T_2 is determined by the equation $g_1 = 1$, i.e. $T_2 = T_c$. In the range $T_2 < T < T_1$, the states with $h \neq 0$ and $h = 0$ correspond to the minimum of the free energy, i.e. one of these states, having larger energy, is metastable. At temperatures $T > T_1$ the paramagnetic state, $h = 0$, is the

only possible solution of the equation of state (18). T_1 is determined by the set of equation (18) and $dG(h)/dh = 1$. The hysteresis region is shown in Figure 1 at different γ . One can see that its width $\Delta T = T_1 - T_2$ increases with increasing γ , however, $\Delta T/T_2 \rightarrow 0$.

The phase diagram of the Potts model on uncorrelated networks was also studied in reference [23] in the framework of a simple mean field approach. The authors of reference [23] found that the first order phase transition is realized when exponent γ is above the threshold line $\gamma(p)$ where $3 < \gamma(p) < 4$. This conclusion was obtained by using a state equation for the average magnetization M , which has a form $M = F(M)$. In principle, this equation resembles our equation of state equation (18). However, the approach used in reference [23] results in a very different behavior of a singular contribution to the state equation. In reference [23], the singular contribution $a_s(\gamma)M^{\gamma-2}$ to $F(M)$ becomes negative, $a_s(\gamma) < 0$, for $\gamma < \gamma(p)$.

In this respect, our results essentially differ from those in reference [23]. In the framework of our approach, the singular contribution to $G(h)$ in equation (18), $g_2^2 h^{\gamma-2}$, is positive for all $\gamma > 3$ and $p \geq 3$. As a result, the phase transition is of the first order in this range of the parameters. The reason for this difference is evidently the simplified mean field theory of reference [23].

6 Continuous transition for $2 < \gamma \leq 3$

The case $\gamma = 3$. – Here, the second moment $\langle k^2 \rangle$ diverges. Using, for brevity, the continuum approximation for the degree distribution, we obtain $G(h) \approx (\langle k \rangle h / (pT)) \ln[p / (\langle k \rangle h)]$. One sees that at small h , the second derivative $d^2G(h)/dh^2$ is negative in contrast to the case $\gamma > 3$ where $d^2G(h)/dh^2$ is positive. It means the change of the order of the phase transition. Instead of the first order phase transition discussed above, the p -state Potts model with $p \geq 3$ undergoes an infinite order phase transition at the critical temperature T_c , equation (1), similarly to the Ising model [11] and percolation [20]. This conclusion also agrees with the phenomenological theory of critical phenomena in complex networks [22].

One should emphasize that when $\langle k^2 \rangle$ diverges, the critical temperature T_c is infinite for the infinite networks (see Eq. (1)). However, in any finite network, $\langle k^2 \rangle < \infty$, and T_c is finite, although it may be very high, $T_c \cong \langle k^2 \rangle / (\langle k \rangle p)$ (see below). At temperatures, which are much less than T_c , but where $h \ll 1$, so $T \gg 1$, we obtain

$$h \cong (p / \langle k \rangle) e^{-2pT / \langle k \rangle}, \quad M \cong e^{-2pT / \langle k \rangle}. \quad (21)$$

Without the continuum approximation, we have, instead of $\langle k \rangle$ in the exponential, a constant which is determined by the complete form of $P(k)$.

The case $2 < \gamma < 3$. – Again T_c for large networks is very high and the phase transition is continuous. Using the expansion $G(h) \cong g(\langle k \rangle / (pT)) h^{\gamma-2}$, we find, in the

range $1 \ll T \ll T_c$, that the Potts model demonstrates the behaviour

$$h, M \sim T^{-1/(3-\gamma)}, \quad (22)$$

which is quite similar to the Ising model [11].

In accordance with equation (1), T_c diverges when $\langle k^2 \rangle \rightarrow \infty$. However, in finite networks, $\langle k^2 \rangle$ is finite because of the finite-size cutoff of the degree distribution. In scale-free networks, it is usually estimated as $k_{cut} \sim k_0 N^{1/(\gamma-1)}$, where N is the total number of vertices in a network, k_0 is a ‘minimal degree’ or the lower boundary of the power-law dependence, and $\langle k \rangle \approx k_0(\gamma-1)/(\gamma-2)$. Then, using estimates from references [4, 20, 32] we obtain

$$T_c \approx \frac{\langle k \rangle \ln N}{p} \quad \text{at } \gamma = 3,$$

$$T_c \approx \frac{4(\gamma-2)^2}{p(3-\gamma)(\gamma-1)} \langle k \rangle N^{(3-\gamma)/(\gamma-1)} \quad \text{for } 2 < \gamma < 3. \quad (23)$$

These expressions generalize the finite-size effect obtained for the Ising model on networks [11].

7 Conclusions

Our investigations of the Potts and Ising models on uncorrelated random networks demonstrate a general strong effect of fat tails in the degree distribution on the order of the phase transition and its critical temperature.

As is natural, at $p = 1$ and $p = 2$, our results coincide with known ones for percolation and the Ising model, respectively. In the Potts model with $p \geq 3$ states, the phase transition of the first order occurs only when the second moment $\langle k^2 \rangle$ is finite, i.e., in scale-free networks, at $\gamma > 3$. When $\langle k^2 \rangle$ diverges, the behaviour of the Potts and Ising models are similar. They undergo the infinite order phase transition. We suggest that this phenomenon (the suppression of a first order phase transition in favour of an infinite order one) is of a general nature and it takes place in other cooperative models with short range interaction on random networks.

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